Math Exam given for the 2025 general admission process at Univ. of Tokyo

Problems for those who wish to major in Science, Engineering, etc. (150 min.)

[1] Let A(0,0), B(0,1), C(1,1), D(1,0) be points on a coordinate plane. Let t satisfy $0<t<1$, P_t, Q_t, R_t be the points on the segments AB,BC,CD, respectively, such that $\dfrac{AP_t}{P_tB}=\dfrac{BQ_t}{Q_tC}=\dfrac{CR_t}{R_tD}=\dfrac{t}{1-t}$,

S_t, T_t be the points on the segments P_tQ_t, Q_tR_t, respectively, such that $\dfrac{P_tS_t}{S_tQ_t}=\dfrac{Q_tT_t}{T_tR_t}=\dfrac{t}{1-t}$,

and U_t be the point on the segment S_tT_t such that $\dfrac{S_tU_t}{U_tT_t}=\dfrac{t}{1-t}$.

Furthermore, let A,D be U_0, U_1, respectively.

(1) Find the coordinates of the point U_t.

(2) Find the area of the domain surrounded by the segment AD and the curve traced by the point U_t, $0 \leqq t \leqq 1$.

(3) Let a satisfy $0<a<1$. Express the length of the curve traced by the point U_t, $0 \leqq t \leqq a$, as a polynomial in a.

[2] (1) Prove $\ln x \leqq x-1$ for $x>0$. (2) Find $\displaystyle\lim_{n\to\infty} n \int_1^2 \ln\left(\frac{1+x^{\frac{1}{n}}}{2}\right)dx$.

[3] A parallelogram ABCD satisfies $\angle ABC=\dfrac{\pi}{6}$, $AB=a$, $BC=b$, and $a \leqq b$.

Consider a rectangle with the condition:

The vertices A,B,C,D lie on the edges EF,FG,GH,HE, respectively, where an edge includes its ends.

Let S be the area of the rectangle EFGH.

(1) Express S in terms of a,b and $\theta=\angle BCG$.

(2) Express the maximum of S in terms of a and b.

[4] A <u>square number</u> is the square of a nonnegative integer.

Let a be a positive integer, and $f_a(x)=x^2+x-a$.

(1) Let n be a positive integer. Prove that $n \leqq a$, if $f_a(n)$ is a square number.

(2) Denote by N_a the number of positive integers n such that $f_a(n)$ is a square number.

Prove that the conditions (i),(ii) below are equivalent:

(i) $N_a=1$ (ii) $4a+1$ is a prime.

[5] There're $n(\geqq 2)$ cards numbered 1 through n, and we arrange them in a row.

Consider the following operation (T_i), where $i=1,2,\cdots$, or $n-1$.

(T_i) If the number of the ith card (from the left end) is greater than that of the $(i+1)$th one, we switch these 2 cards. Otherwise, we do nothing.

Suppose that the number of the ith card is A_i $(1 \leqq i \leqq n)$ in the beginning, and it turns i for $i=1,\cdots,n$ by $(n-1)$ operations $(T_1),(T_2),\cdots,(T_{n-1})$ followed by $(n-1)$ operations $(T_{n-1}),\cdots,(T_2),(T_1)$.

(1) Prove that at least one of A_1, A_2 is not greater than 2.

(2) Let C_n be the number of possible arrangement $A_1 \cdots A_n$.

For $n \geqq 4$, express C_n in terms of C_{n-1} and C_{n-2}.

[6] On a plane of complex numbers, let C be the circle centered at $\dfrac{1}{2}$ with radius $\dfrac{1}{2}$, minus zero.

(1) For $z \in C$, prove that the real part of $\dfrac{1}{z}$ is 1.

(2) If $\alpha, \beta \in C$ and they're distinct, express the domain in which $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ moves around.

(3) If γ is a complex number belonging to the complement of the domain in (2), find the maximum and the minimum of the real part of $\dfrac{1}{\gamma}$.